

MINIMAL AND MAXIMAL OPERATOR SPACE STRUCTURES ON BANACH SPACES

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ABSTRACT. Given a Banach space X , there are many operator space structures possible on X , which all have X as their first matrix level. Blecher and Paulsen [4] identified two extreme operator space structures on X , namely $Min(X)$ and $Max(X)$ which represents respectively, the smallest and the largest operator space structures admissible on X . In this note, we consider the subspace and the quotient space structure of minimal and maximal operator spaces.

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1. INTRODUCTION

Operator spaces form a natural quantization of Banach spaces and their study took a rigorous form with the representation theorem obtained by Z. J. Ruan [17] in 1988 and after that it has seen considerable development with applications to the theory of operator algebras and various aspects of operator spaces are being studied extensively.

A *concrete operator space* X is a closed linear subspace of $B(\mathcal{H})$, for some Hilbert space \mathcal{H} . Here, in each matrix level $M_n(X)$, we have a norm $\|\cdot\|_n$, induced by the inclusion $M_n(X) \subset M_n(B(\mathcal{H}))$, where the norm in $M_n(B(\mathcal{H}))$ is given by the natural identification $M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n)$. More precisely, for $[x_{ij}] \in M_n(X)$, we have

$$\|[x_{ij}]\|_n = \sup\{(\sum_{i=1}^n \|\sum_{j=1}^n x_{ij} h_j\|_{\mathcal{H}}^2)^{1/2} \mid h_j \in \mathcal{H}, (\sum \|h_j\|_{\mathcal{H}}^2)^{1/2} \leq 1\}.$$

Thus, a concrete operator space carries not just an inherited norm, but these additional sequence of matrix norms.

An *abstract operator space*, or simply an *operator space* is a pair $(X, \{\|\cdot\|_n\}_{n \in \mathbb{N}})$ consisting of a linear space X and a complete norm $\|\cdot\|_n$ on $M_n(X)$ for every $n \in \mathbb{N}$, such that there exists a linear complete isometry $\varphi : X \rightarrow B(\mathcal{H})$ for

some Hilbert space \mathcal{H} . The sequence of matrix norms $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ is called an *operator space structure* on the linear space X . An operator space structure $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ on a Banach space $(X, \|\cdot\|)$ is said to be an *admissible* operator space structure on X , if $\|\cdot\|_1 = \|\cdot\|$. An important type of operator spaces are those $X \subset B(\mathcal{H})$ which are isometric (as a Banach space) to a Hilbert space. Such spaces are called *Hilbertian* operator spaces [13].

If X is a Banach space, then any linear isometry from X to $B(\mathcal{H})$, for some Hilbert space \mathcal{H} , endows an operator space structure on X . Generally, for a Banach space X , the matrix norms so obtained are not unique. In other words, a given Banach space has, in general, many realizations as an operator space. Blecher and Paulsen observed that the set of all operator space structures admissible on a given Banach space X admits a minimal and a maximal element. The minimal and the maximal operator space structures on a Banach space were introduced and their dual relations were explored in [4] and further structural properties were investigated in [10] and [11]. In what follows, we focus on the subspace and quotient space structure of minimal and maximal operator spaces.

It is known that any subspace of a minimal operator space is again minimal, but quotient of a minimal space need not be minimal. A subspace of a maximal operator space need not be maximal. But quotient spaces inherits the maximality. We address the following question: If every proper, nontrivial subspace of an operator space X is minimal (maximal), is X minimal? (maximal?). We give an example to show that the answer is negative in the case of finite dimensional operator spaces, and show that the answer is affirmative in the case of infinite dimensional operator spaces.

Regarding quotient operator spaces, we prove that, if X is an infinite dimensional operator space, and if every quotient of X by a proper closed nontrivial subspace of X is minimal (maximal), then X is minimal (maximal). We also give an example to show that the result is invalid in the case of finite dimensional operator spaces.

2. MINIMAL AND MAXIMAL OPERATOR SPACES

Let X be a Banach space and X^* be its dual space. Let $K = \text{Ball}(X^*)$ be the closed unit ball of the dual space of X with its weak* topology. Then the canonical embedding $J : X \rightarrow C(K)$, defined by $J(x)(f) = f(x)$, $x \in X$ and

$f \in K$ is a linear isometry. Since, subspaces of C^* -algebras are operator spaces (by Gelfand-Naimark Theorem), this identification of X induces matrix norms on $M_n(X)$ that makes X an operator space and the matrix norms on X are given by

$$\|[x_{ij}]\|_n = \sup\{\|[f(x_{ij})]\| \mid f \in K\}$$

for all $[x_{ij}] \in M_n(X)$ and for all $n \in \mathbb{N}$.

Here $\|[f(x_{ij})]\|$ indicates the norm of the scalar $n \times n$ matrix $[f(x_{ij})]$ viewed as a linear map from $\mathbb{C}^n \rightarrow \mathbb{C}^n$.

The above defined operator space structure on X is called the *minimal operator space structure* on X , and we denote this operator space as $Min(X)$. Thus, $Min(X)$ can be regarded as a space of continuous functions defined on the closed unit ball of X^* . An operator space X is said to be *minimal* if $Min(X) = X$. The minimal operator space structure of a Banach space is characterized by the universal property that for any arbitrary operator space Y , any bounded linear map $\varphi : Y \rightarrow Min(X)$ is completely bounded and satisfies $\|\varphi : Y \rightarrow Min(X)\|_{cb} = \|\varphi : Y \rightarrow X\|$. The above described universal property implies that $Min(X)$ is indeed the smallest admissible operator space structure on a Banach space X . For, if $\{\|\cdot\|'_n\}_{n \in \mathbb{N}}$ is any other admissible operator space structure on X , and if \tilde{X} denotes the space X with these matrix norms, then $id : \tilde{X} \rightarrow Min(X)$ is a linear isometry and $\|id\|_{cb} = \|id\| = 1$. This shows that $\|\cdot\|'_n$ dominates the corresponding matrix norms in $Min(X)$. It is known that, an operator space is minimal if and only if it is completely isometric to a subspace of a commutative C^* -algebra [5].

If X is a Banach space, there is a *maximal* way to consider it as an operator space. For $[x_{ij}] \in M_n(X)$, the matrix norms given by

$$\|[x_{ij}]\| = \sup \|\varphi(x_{ij})\|$$

where the supremum is taken over all operator spaces Y and all linear maps $\varphi \in Ball(B(X, Y))$, define an admissible operator space structure on X . We denote this operator space as $Max(X)$ and is called the *maximal operator space structure* on X . An operator space X is said to be *maximal* if $Max(X) = X$. By the definition of $Max(X)$, any operator space structure that we can put on X , must be smaller than $Max(X)$.

The maximal operator space structure of a Banach space is characterized by the universal property that for any arbitrary operator space Y , any bounded linear map $\varphi : Max(X) \rightarrow Y$ is completely bounded and satisfies

$$\|\varphi : Max(X) \rightarrow Y\|_{cb} = \|\varphi : X \rightarrow Y\|.$$

Thus, if X and Y are Banach spaces and $\varphi \in B(X, Y)$, then φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$, when considered as a map from $Max(X) \rightarrow Y$.

To see that the space $Max(X)$ satisfies the above mentioned property, let $\varphi : Max(X) \rightarrow Y$ be a bounded linear map. Then $u = \frac{\varphi}{\|\varphi\|} \in Ball(X, Y)$, and by definition of the matrix norms of $Max(X)$, $\|[u(x_{ij})]\|$ is dominated by $\|[x_{ij}]\|_{M_n(Max(X))}$, for all $[x_{ij}] \in M_n(X)$ and for all $n \in \mathbb{N}$, showing that $\|u\|_{cb} \leq 1$. Thus $\|\varphi\|_{cb} \leq \|\varphi\|$. Therefore, $\varphi : Max(X) \rightarrow Y$ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$.

The above described universal property implies that $Max(X)$ is indeed the largest admissible operator space structure on a Banach space X . For, if $\{\|\cdot\|'_n\}_{n \in \mathbb{N}}$ is any other admissible operator space structure on X , and if \tilde{X} denotes the space X with these matrix norms, then $id : Max(X) \rightarrow \tilde{X}$ is a linear isometry and $\|id\|_{cb} = \|id\| = 1$. This shows that $\|\cdot\|'_n$ is dominated by the corresponding matrix norms in $Max(X)$.

The following proposition gives characterizations of minimal and maximal operator spaces up to complete *isomorphisms*. These characterizations identify larger classes of operator space structures which are completely isomorphic (need not be completely isometric) to minimal and to maximal operator spaces.

Proposition 2.1.

- (i). An operator space X is completely isomorphic to a minimal operator space if and only if for any arbitrary operator space Y , any completely bounded linear bijection $\varphi : X \rightarrow Y$ is a complete isomorphism.
- (ii). An operator space X is completely isomorphic to a maximal operator space if and only if for any arbitrary operator space Y , any completely bounded linear bijection $\varphi : Y \rightarrow X$ is complete isomorphism.

Proof.

We prove only (i) and (ii) will follow in a similar way. Assume that $\varphi : X \rightarrow Y$ is a completely bounded linear bijection. Let $\psi : X \rightarrow Min(Z)$ be a complete isomorphism. Then by the universal property of minimal operator spaces, $\psi \circ \varphi^{-1} : Y \rightarrow Min(Z)$ is completely bounded. Therefore, $\|\varphi^{-1}\|_{cb} = \|\psi^{-1} \circ \psi \circ \varphi^{-1}\|_{cb} \leq \|\psi^{-1}\|_{cb} \|\psi \circ \varphi^{-1}\|_{cb} < \infty$. This shows that $\varphi : X \rightarrow Y$ is a complete isomorphism. For the converse, take $Y = Min(X)$

and consider the formal identity mapping $id : X \rightarrow Min(X)$. By assumption, $id^{-1} : Min(X) \rightarrow X$ is completely bounded, showing that X is completely isomorphic to $Min(X)$. \square

Remark 2.2.

The above theorem describes the complete isomorphism class of minimal and maximal operator spaces. Recently, T. Oikhberg [9] proved that the complete isomorphism class of any infinite dimensional operator space has infinite diameter with respect to the completely bounded Banach-Mazur distance. i.e., for $n \in \mathbb{N}$, $\forall C > 0$, and for any infinite dimensional operator space X , there exists an operator space structure \tilde{X} on X such that the identity map $id : X \rightarrow \tilde{X}$ is a complete isomorphism, $id^{(n)}$ is an isometry, and $d_{cb}(X, \tilde{X}) > C$. Measuring the diameter of the complete isomorphism class of an operator space is still open in the case of finite dimensional operator spaces.

Let X be an infinite dimensional Banach space. Then operator space structures on X , which are completely isomorphic to $Min(X)$ can be constructed as follows: Choose an operator space Y which is isometric to X and completely isomorphic to $Min(X)$, say $v : Min(X) \rightarrow Y$ be a complete isomorphism (From the above remark, such a choice is always possible). On X , define a new operator space structure, say \tilde{X} , by setting $\|[x_{ij}]\|_{M_n(\tilde{X})} = \|[v(x_{ij})]\|_{M_n(Y)}$, $\forall [x_{ij}] \in M_n(X)$ and $\forall n \in \mathbb{N}$. Then, \tilde{X} and Y are completely isometrically isomorphic, and so $Min(X)$ and \tilde{X} are completely isomorphic. In a similar way, we can construct operator space structures on X which are completely isomorphic to $Max(X)$.

The following theorem describes the dual nature of minimal and maximal operator space structures on a Banach space.

Theorem 2.3 ([1]). *For any Banach space X , we have $Min(X)^* \cong Max(X^*)$ and $Max(X)^* \cong Min(X^*)$ completely isometrically.*

3. SUBMAXIMAL SPACES AND Q -SPACES

From the definition of minimal operator spaces, it is clear that any subspace of a minimal operator space is again minimal, but a quotient of a minimal space need not be minimal. An operator space that is a quotient of a minimal operator space (up to complete isometric isomorphism) is called a Q -space [14]. Since an operator space is minimal if and only if it is completely isometric to a

subspace of a commutative C^* -algebra [5], Q -spaces are precisely the quotients of subspaces of commutative C^* -algebras. Also, the category of Q -spaces is stable under taking quotients and subspaces. Q -spaces were investigated by M. Junge [6] and by Blecher and Le Merdy [2].

Q -spaces need not be minimal, for instance, the space $R \cap C$ is a Q -space, as it can be identified with the quotient space $L^\infty[0, 1]/S$, where S is the subspace orthogonal to the Rademacher functions [12]. But, $R \cap C$ is not minimal, and moreover $d_{cb}(R_n \cap C_n, \text{Min}(\ell_2^n)) = \sqrt{n}$ [13], so that $R \cap C$ is not completely isomorphic to $\text{Min}(\ell_2)$.

Another example for a Q -space, which is not minimal is furnished by the space of Hankel matrices that can be identified with L^∞/H^∞ . It can be shown that it has a subspace which is completely isometric to $R \cap C$, so that the space L^∞/H^∞ is not minimal [12].

Subspace structure of various maximal operator spaces were studied in [8]. Subspaces of maximal operator spaces are called *submaximal* spaces and in general they need not be maximal, i.e., if Y is a subspace of X and if $x_{ij} \in Y$ for $i, j = 1, 2, \dots, n$, then the norm of $[x_{ij}]$ in $M_n(\text{Max}(Y))$ can be larger than the norm of $[x_{ij}]$ as an element of $M_n(\text{Max}(X))$. For example, the space $R + C$ is submaximal, as it can be identified as a closed subspace of $\text{Max}(L_1)$ spanned by the Rademacher functions [7]. But $R + C$ is not maximal and moreover $d_{cb}(R_n + C_n, \text{Max}(\ell_2^n)) = \sqrt{n}$ [13], so that $R + C$ is not completely isomorphic to $\text{Max}(\ell_2)$. However, Paulsen obtained the following result.

Theorem 3.1 ([11]). *Let X be an infinite dimensional operator space and $x_{ij} \in X$, for $i, j = 1, 2, \dots, n$, then*

$$\|[x_{ij}]\|_{M_n(\text{Max}(X))} = \inf \{ \|[x_{ij}]\|_{M_n(\text{Max}(Y))} \mid x_{ij} \in Y, Y \subset X, \text{finite dimensional} \}$$

But quotient spaces inherits the maximality as illustrated in the following theorem [13].

Theorem 3.2 ([13]). *If X is a maximal operator space and Y is a closed subspace of X , then $\text{Max}(X/Y) \cong \text{Max}(X)/Y$ completely isometrically.*

Also, if every subspace of $\text{Max}(X)$ is maximal, then any two Banach isomorphic subspaces of X will be completely isomorphic as subspaces of $\text{Max}(X)$. For, if E and F are Banach isomorphic subspaces of X , then E and F are completely isomorphic as subspaces of $\text{Max}(X)$. In [15], the notion of hereditarily maximal spaces is introduced. Hereditarily maximal spaces determine a

subclass of maximal operator spaces with the property that the operator space structure induced on any subspace coincides with the maximal operator space structure on that subspace. Also, it is proved that the class of hereditarily maximal spaces includes all Hilbertian maximal operator spaces. Since ℓ_1^2 has a unique operator space structure, ℓ_1^2 is a maximal operator space and all of its subspaces are maximal. So, ℓ_1^2 is an example for a hereditarily maximal space which is not Hilbertian. The smallest submaximal space structure $\mu(X)$, admissible on an operator space X is studied in [8] and [16].

The following result reveals the natural duality between subspaces and quotient spaces.

Theorem 3.3 ([5]). *If Y is a closed subspace of an operator space X , then, $(X/Y)^* \cong Y^\perp$ and $Y^* \cong X^*/Y^\perp$ completely isometrically, where*

$$Y^\perp = \{f \in X^* \mid f(y) = 0, \forall y \in Y\}.$$

Let Y be a submaximal space, say $Y \subset \text{Max}(X)$. Then by Theorem 3.3, $Y^* \cong (\text{Max}(X))^*/Y^\perp$. But by using Theorem 2.3, we get $(\text{Max}(X))^*/Y^\perp \cong \text{Min}(X^*)/Y^\perp$ showing that the dual of a submaximal space is a Q -space.

Conversely, if $Z = \text{Min}(X)/Y$ is a Q -space, then by Theorem 3.3, $Z^* = (\text{Min}(X)/Y)^* \cong Y^\perp$. But here,

$$\begin{aligned} Y^\perp &= \{f \in (\text{Min}(X))^* \mid f(y) = 0, \forall y \in Y\} \\ &= \{f \in \text{Max}(X^*) \mid f(y) = 0, \forall y \in Y\} \end{aligned}$$

so that, Z^* is a submaximal space. Thus, the dual of a submaximal space is a Q -space and vice versa.

We have noted that Q -spaces need not be minimal. But, using Theorem 3.3, we observe that if X is a Hilbertian operator space, then any Q -space in X (i.e., any quotient of $\text{Min}(X)$) is minimal.

Let X be a Hilbertian operator space and Y be a closed subspace of X . Let $id : \text{Min}(Y^\perp) \rightarrow \text{Min}(X)$ be the formal identity mapping, and

$\pi : \text{Min}(X) \rightarrow \text{Min}(X)/Y$ be the quotient map.

Then $\pi \circ id : \text{Min}(Y^\perp) \rightarrow \text{Min}(X)/Y$ is a complete contraction. Since, X/Y is isometric to Y^\perp , $\text{Min}(X/Y)$ is completely isometric to $\text{Min}(Y^\perp)$, so that

$$\|[x_{ij} + Y]\|_{M_n(\text{Min}(X)/Y)} \leq \|[x_{ij} + Y]\|_{M_n(\text{Min}(X/Y))}.$$

But, by definition of minimal operator spaces,

$$\|[x_{ij} + Y]\|_{M_n(\text{Min}(X/Y))} \leq \|[x_{ij} + Y]\|_{M_n(\text{Min}(X)/Y)}.$$

Thus, $\text{Min}(X)/Y$ is completely isometrically isomorphic to $\text{Min}(X/Y)$.

Remark 3.4.

There are non-Hilbertian operator spaces X for which all Q -spaces in X are minimal. For instance, ℓ_1^2 is a minimal operator space, which is not Hilbertian and all Q -spaces in ℓ_1^2 are minimal. Identification of a subclass of minimal operator spaces with the property that the operator space structure induced on any quotient space coincides with the minimal operator space structure on that quotient space is still open.

4. MAIN RESULTS

We have noted that a subspace of a minimal operator space is minimal, whereas a subspace of a maximal space need not be maximal. On the other hand, let us consider the following question: If every proper, nontrivial subspace of an operator space X is minimal (maximal), is X minimal? (maximal?) The answer to this question is No. For instance, if X is of dimension 2, and if \tilde{X} is any operator space structure on X such that $\tilde{X} \neq \text{Min}(X)$ (resp. $\tilde{X} \neq \text{Max}(X)$) (such a space exists since any Banach space of dimension greater than 2 has more than one quantization, i.e., the space has more than one admissible operator space structure [13].) Then any proper, nontrivial subspace of \tilde{X} will be of dimension 1, and so is minimal (resp. maximal). (Since there is only one operator space of dimension 1, up to complete isometric isomorphism.)

But in the case of infinite dimensional operator spaces, we have an affirmative answer.

Theorem 4.1. *Let X be an infinite dimensional operator space. If every finite dimensional subspace of X is minimal (maximal), then X is minimal (maximal).*

Proof.

First, assume that every finite dimensional subspace of X is minimal. Let $[x_{ij}] \in M_n(X)$. Then, $[x_{ij}] \in M_n(E)$, where $E = \text{span} \{x_{ij}\}$ and the dimension of $E \leq n^2$. By assumption, E is minimal and by using the Hahn-Banach

theorem, we have

$$\begin{aligned}
 \|[x_{ij}]\|_{M_n(X)} &= \|[x_{ij}]\|_{M_n(E)} \\
 &= \|[x_{ij}]\|_{M_n(\text{Min}(E))} \\
 &= \sup\{\|[f(x_{ij})]\| \mid f \in \text{Ball}(E^*)\} \\
 &\leq \sup\{\|[f(x_{ij})]\| \mid f \in \text{Ball}(X^*)\} \\
 &= \|[x_{ij}]\|_{M_n(\text{Min}(X))}
 \end{aligned}$$

Since $\text{Min}(X)$ is the smallest operator space structure on X , this shows that $\|[x_{ij}]\|_{M_n(X)} = \|[x_{ij}]\|_{M_n(\text{Min}(X))}$ and hence X is minimal.

Now to prove the maximal case, by Theorem 3.1, for any $[x_{ij}] \in M_n(X)$, we have

$$\begin{aligned}
 \|[x_{ij}]\|_{M_n(\text{Max}(X))} &= \inf\{\|[x_{ij}]\|_{M_n(\text{Max}(Y))} \mid x_{ij} \in Y, Y \subset X, \text{ finite dimensional}\} \\
 &= \inf\{\|[x_{ij}]\|_{M_n(Y)} \mid x_{ij} \in Y, Y \subset X, \text{ finite dimensional}\} \\
 &\leq \|[x_{ij}]\|_{M_n(X)}
 \end{aligned}$$

Since $\text{Max}(X)$ is the largest operator space structure on X , this shows that $\|[x_{ij}]\|_{M_n(\text{Max}(X))} = \|[x_{ij}]\|_{M_n(X)}$ and hence X is maximal. \square

We have noted that quotients of minimal operator spaces need not be minimal, whereas quotients of maximal operator spaces are maximal. In the case of quotient spaces of an infinite dimensional operator space, we have the following result.

Theorem 4.2. *Let X be an infinite dimensional operator space.*

- (i). *If every quotient of X by a proper closed nontrivial subspace of X is minimal, then X is minimal.*
- (ii). *If every quotient of X by a proper closed nontrivial subspace of X is maximal, then X is maximal.*

For proving this, we make use of the following theorems.

Theorem 4.3 ([5]). *If X is any operator space and $[x_{ij}] \in M_n(X)$, there exists a complete contraction $\varphi : X \rightarrow M_n$ such that $\|\varphi^{(n)}([x_{ij}])\| = \|[x_{ij}]\|$.*

Theorem 4.4 ([3]). *Let X and Z be operator spaces. If $\phi : X \rightarrow Z$ is completely bounded, and if Y is a closed subspace of X contained in $\ker(\phi)$, then the canonical map $\tilde{\varphi} : X/Y \rightarrow Z$ induced by φ is also completely bounded,*

with $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$. If $Y = \ker(\varphi)$, then φ is a complete quotient map if and only if $\tilde{\varphi}$ is a completely isometric isomorphism.

Proof of Theorem 4.2.

We first prove the second part. Let $[x_{ij}] \in M_n(X)$. By Theorem 4.3, we have

$$(1) \quad \|[x_{ij}]\|_{M_n(X)} = \sup\{\|\varphi(x_{ij})\| \mid \varphi : X \rightarrow M_n, \|\varphi\|_{cb} \leq 1\}$$

Since X is infinite dimensional, and the range of φ is finite dimensional, φ has a nontrivial kernel, $\ker(\varphi)$. Let $\tilde{\varphi} : X/\ker(\varphi) \rightarrow M_n$ be the canonical map defined by $\tilde{\varphi}(x + \ker(\varphi)) = \varphi(x)$. Then by Theorem 4.4, $\|\varphi\|_{cb} = \|\tilde{\varphi}\|_{cb} = \|\tilde{\varphi}\| = \|\varphi\|$, where the last but one equality follows from the assumption that $X/\ker(\varphi)$ is maximal. Thus in equation (1), we can replace $\|\varphi\|_{cb} \leq 1$ by $\|\varphi\| \leq 1$, so that we have

$$(2) \quad \|[x_{ij}]\|_{M_n(X)} = \sup\{\|\varphi(x_{ij})\| \mid \varphi : X \rightarrow M_n, \|\varphi\| \leq 1\}$$

Applying equation (1) to the operator space $Max(X)$, and using the universal property of $Max(X)$ and the equation (2), we obtain

$$\begin{aligned} \|[x_{ij}]\|_{M_n(Max(X))} &= \sup\{\|\varphi(x_{ij})\| \mid \varphi : Max(X) \rightarrow M_n, \|\varphi\|_{cb} \leq 1\} \\ &= \sup\{\|\varphi(x_{ij})\| \mid \varphi : Max(X) \rightarrow M_n, \|\varphi\| \leq 1\} \\ &= \sup\{\|\varphi(x_{ij})\| \mid \varphi : X \rightarrow M_n, \|\varphi\| \leq 1\} \\ &= \|[x_{ij}]\|_{M_n(X)} \end{aligned}$$

This shows that $X = Max(X)$ and so X is maximal.

For proving the first part, we note that for any $[x_{ij}] \in M_n(X)$,

$$(3) \quad \|[x_{ij}]\|_{M_n(X)} = \sup\{\|[x_{ij} + Y]\|_{M_n(X/Y)} \mid Y \subset X\}$$

Now, we assume that every quotient of X by a proper closed nontrivial subspace of X is minimal.

Let $[x_{ij}] \in M_n(X)$. We claim that

$$\|[x_{ij}]\|_{M_n(X)} = \sup\{\|[x_{ij} + Y]\|_{M_n(X/Y)} \mid Y \subset X, \dim(X/Y) < \infty\}$$

From equation (3), we have

$$\|[x_{ij}]\|_{M_n(X)} \geq \sup\{\|[x_{ij} + Y]\|_{M_n(X/Y)} \mid Y \subset X, \dim(X/Y) < \infty\}$$

By Theorem 4.3, there exists a complete contraction $\varphi : X \rightarrow M_n$ such that $\|\varphi^{(n)}([x_{ij}])\| = \|[x_{ij}]\|_{M_n(X)}$. Let $Y = \ker(\varphi)$, then X/Y is finite dimensional and let $\tilde{\varphi}$ be the canonical map. Then,

$$\|[x_{ij}]\|_{M_n(X)} = \|\varphi^{(n)}([x_{ij}])\| = \|\tilde{\varphi}^{(n)}([x_{ij} + Y])\| \leq \|[x_{ij} + Y]\|_{M_n(X/Y)}.$$

This implies,

$$\|[x_{ij}]\|_{M_n(X)} \leq \sup\{\|[x_{ij} + Y]\|_{M_n(X/Y)} \mid Y \subset X, \dim(X/Y) < \infty\}$$

This proves the claim.

Now let $id : Min(X) \rightarrow X$ be the formal identity map. Let Y be a subspace of X such that $\dim(X/Y) < \infty$ and $\pi : X \rightarrow X/Y$ be the quotient mapping. Then, $\tilde{id} = \pi \circ id : Min(X) \rightarrow X/Y$ is given by $\tilde{id}(x) = x + Y$. Since, by assumption, X/Y is minimal and by using the universal property of minimal spaces, \tilde{id} is completely bounded and $\|\tilde{id}\|_{cb} = \|\tilde{id}\| \leq \|\pi\| \|id\| \leq 1$. This shows that \tilde{id} is completely contractive. Therefore,

$$\begin{aligned} \|[x_{ij}]\|_{M_n(X)} &= \sup\{\|[x_{ij} + Y]\|_{M_n(X/Y)} \mid Y \subset X, \dim(X/Y) < \infty\} \\ &= \sup\{\|\tilde{id}(x_{ij})\|_{M_n(X/Y)} \mid Y \subset X, \dim(X/Y) < \infty\} \\ &\leq \|[x_{ij}]\|_{M_n(Min(X))}. \end{aligned}$$

This shows that $X = Min(X)$, and so X is minimal. \square

Remark 4.5.

If X is finite dimensional, the above result need not be true. For instance, if X is an operator space of dimension 2 which is not minimal (maximal), then every quotient of X by a proper, nontrivial closed subspace will be of dimension 1 and hence is minimal (maximal).

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